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THEORY OF STEADY WAVES IN HORIZONTAL FLOW WITH A LINEAR VELOCITY PROFILE

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The structure and characteristics of nonlinear steady waves on the surface of horizontal shear flow of an ideal homogeneous incompressible fluid of finite depth with a linear velocity profile are studied using two-dimensional theory and the Euler approach. The wave motion is considered irrotational. A modification of the first Stokes method is proposed that allows algebraic calculations of terms of perturbation series. Nonlinear dispersion relations are obtained and analyzed for both upstream and downstream traveling waves.

Key words: *steady nonlinear waves, ideal fluid, shear flow, first Stokes method, nonlinear dispersion relations.*

Introduction. Nonlinear, in particular, steady waves in shear flows have been studied in many papers (see, for example, [1] and the bibliography therein). However, nonlinear dispersion relations have not been derived and analyzed even for the simplest case of waves in flows with a linear velocity profile. In [1], such a relation is given in the form of a lengthy integral formula for a more general case but its application is an even more difficult problem than the original one.

The goal of the present paper is to study two-dimensional wave motion of a finite-depth fluid with a linear mean-velocity profile, which, as in the absence of mean flow (i.e., in the Stokes problem), admits the existence of irrotational wave motion becomes possible. The problem is solved using a modification of the first Stokes method [2–5]. A feature of our approach is that the two-dimensional problem is reduced to a one-dimensional problem. This procedure is simplified by introducing auxiliary functions. In addition, perturbation series similar to Stokes series are used. Linear equations were obtained and solved for the lower approximations. This problem was considered previously in [6], where considerable attention was paid to transformations of nonlinear boundary conditions whereas a description of the solution technique is almost absent, which makes its analysis impossible. In addition, the fact that there are two types of waves of different structures — upstream and downstream traveling waves — is ignored in [6]. Therefore, the given problem requires a new solution.

Formulation of the Problem. We consider horizontal flow of an ideal incompressible homogeneous fluid of finite depth h with a linear mean-velocity profile: $\bar{u} = by$, $\bar{v} = 0$, and $b = \text{const}$. We assume that a system of steady-state nonlinear waves moving at constant velocity c was formed on the free surface. We use a rectangular coordinate system (x, y) with the x axis coincident with the mean horizontal level and the axis y directed upright. The horizontal and vertical components of the fluid particle velocity will be denoted by $\bar{u} + u$ and v, where u and v are the values of these components due to wave motion. The pressure, density, and profile of the free surface will be denoted by p, ρ and η, respectively. In the case of steady-state waves, the dynamic variables depend on the coordinates and time as follows: $\eta = \eta(x - ct)$ and $(u, v, p) = (u, v, p)(x - ct, y)$. The choice of the horizontal axis leads to the zero mean condition for the wave profile: $\langle \eta(x) \rangle = 0$.

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For an ideal incompressible homogeneous fluid, the Euler dynamic equations supplemented by the condition of potentiality of wave motion have the form

$$\rho((u+by-c)u_x + v(b+u_y)) + p_x = 0,$$
$$\rho((u+by-c)v_x + vv_y + g) + p_y = 0; \tag{1}$$

$$u_x + v_y = 0, \qquad u_y - v_x = 0. \tag{2}$$

Here and below, $\eta = \eta(x)$ and $(u, v, p) = (u, v, p)(x, y)$, i.e., we set $x - ct \to x$.

On the free surface $y = \eta(x)$, two boundary conditions hold: $(u^s + b\eta - c)\eta' - v^s = 0$ and $p^s = 0$; here and below, the superscript s indicates that the values are taken on the free surface, for example, $p^s = p(x, \eta(x))$. The boundary condition on the bottom is the nonpenetration condition $v(x, -h) = 0$.

In addition, we adopt the periodicity conditions

$$\eta(x+L) = \eta(x), \qquad u(x+L, y) = u(x, y), \qquad v(x+L, y) = v(x, y)$$

(L is the wavelength) and the zero mean (on x) condition: $\langle \eta(x) \rangle = 0$.

The stream function $\psi = \psi(x, y)$ is defined by the equalities $u = \psi_y$ and $v = -\psi_x$. Then, Eqs. (2) reduce to the Laplace equation

$$\Delta\psi = \psi_{xx} + \psi_{yy} = 0.$$

The stream function also satisfies the periodicity and zero mean conditions.

Equations (1) and (2) admit the first integral, which is expressed in terms of the stream function as

$$P = \rho(-V(x, y) + b(y\psi_y - \psi) + gy) + p = \text{const},$$

where $V(x, y) = c\psi_y - 2^{-1}(\psi_x^2 + \psi_y^2)$. This integral is an extension of the Bernoulli integral to the case of constant vorticity flows. The quantity $\Psi(x, y) = \psi(x, y) - cy + 2^{-1}by^2$ is the total stream function in a coordinate system moving together with the wave. In this coordinate system, the motion will be steady-state; therefore, on the free surface, the quantity Ψ takes a constant value:

$$\Psi^s = \psi^s(x) - c\eta(x) + 2^{-1}b\eta^2(x) = Q = \text{const}.$$

By virtue of the aforesaid, for the boundary conditions we have

$$-c\eta(x) + \psi^s(x) + 2^{-1}b\eta^2(x) = Q,$$
$$-V^s(x) + b(\eta(x)\psi_y(x, \eta(x)) - \psi^s(x)) + g\eta(x) = P, \qquad P = \text{const}, \tag{3}$$
$$\psi(x, -h) = 0.$$

It is easy to obtain the solution of the problem considered in a linear approximation:

$$\eta(x) = a\cos(kx), \qquad \psi(x, y) = \sinh^{-1}(kh)\, c_0 a \cos(kx) \sinh(k(y+h)), \qquad k = 2\pi/L.$$

Here c_0 is the phase velocity of linear waves, which satisfies the equation

$$k \coth(kh) c_0^2 + bc_0 - g = 0.$$

This equation has two real roots with opposite signs. They correspond to two sine waves traveling downstream and upstream.

Reduction to a One-Dimensional Problem. We introduce new auxiliary functions $\psi(x) = \psi(x, 0)$, $\xi(x) = \psi_y(x, 0)$, $H(x) = \eta^2(x)$, and $V(x) = V_y(x, 0)$, which will be used to pass from the original two-dimensional problem to a one-dimensional problem and to simplify the procedure of calculating successive approximations. We make the following remarks.

1. The notation $\psi(x)$ coincides with the notation of the stream function, but this should not lead to a confusion since the stream function will not be considered below.

2. The correspondence $\psi(x) \to \xi(x)$ is a linear operation W that allows one to uniquely determine $\xi(x)$ on $\psi(x)$: $\xi(x) = W\psi(x)$; in particular, for $\psi(x) = \cos(kx)$, we obtain $\xi(x) = k\coth(kh)\cos(kx)$. By virtue of linearity, from this particular case, it is easy to obtain the value $\xi(x)$ for any trigonometric polynomial.

Using the new functions and the operator W, we obtain the equations of the one-dimensional problem (with accuracy up to the 3rd approximation)

$$\begin{gathered}\xi(x) = W\psi(x),\\ g\eta(x) - b\psi(x) - c\xi(x) + 2^{-1}(\xi^2(x) + (\psi'(x))^2) - \eta(x)V(x) + 2^{-1}cH(x)\xi''(x) = P,\\ -c\eta(x) + \psi(x) + \eta(x)\xi(x) + 2^{-1}H(x)(b - \psi''(x)) = Q,\\ H(x) = \eta^2(x), \qquad V(x) = (\xi(x) - c)\psi''(x) - \psi'(x)\xi'(x).\end{gathered} \tag{4}$$

In the derivation of these equations, we used the Taylor expansion of Eqs. (3) in series in powers of $\eta(x)$.

Derivation and Solution of Systems of Equations of Successive Approximations. An approximate solution of the one-dimensional problem is sought in the form

$$c = c_0(1 + c_1(ka)^2), \qquad Q = k^{-1}c_0Q_1(ka)^2, \qquad P = k^{-1}c_0P_1(ka)^2,$$

$$\eta(x) = k^{-1}(\eta_1(x)(ka) + \eta_2(x)(ka)^2 + \eta_3(x)(ka)^3), \qquad \eta_1(x) = \cos(kx),$$

$$\psi(x) = k^{-1}c_0(\psi_1(x)(ka) + \psi_2(x)(ka)^2 + \psi_3(x)(ka)^3),$$

$$\xi(x) = c_0(\xi_1(x)(ka) + \xi_2(x)(ka)^2 + \xi_3(x)(ka)^3),$$

$$H(x) = k^{-2}H_2(x)(ka)^2 + k^{-2}H_3(x)(ka)^3,$$

$$V(x) = kc_0^2(V_1(x)(ka) + V_2(x)(ka)^2)$$

(a is the amplitude of the fundamental harmonic in the wave profile). Here allowance was made for the solution of the linear problem. Substituting these relations into Eqs. (4) and splitting the result into powers of ka, we obtain the systems of the three lowest approximations:

— the system of equations of the 1st approximation

$$\xi_1(x) = k^{-1}W\psi_1(x), \qquad (Rkc_0 + b)\eta_1(x) - b\psi_1(x) - kc_0\xi_1(x) = 0,$$

$$-\eta_1(x) + \psi_1(x) = 0, \qquad \eta_1(x) = \cos(kx), \qquad R = \coth(kh);$$

$$\xi_2(x) = k^{-1}W\psi_2(x), \qquad (Rkc_0 + b)\eta_2(x) - b\psi_2(x) - kc_0\xi_2(x) = A_2(x) + P_1,$$

$$-\eta_2(x) + \psi_2(x) = B_2(x) + Q_1, \qquad H_2(x) = \eta_1^2(x), \qquad V_1(x) = -k^{-2}\psi_1''(x),$$

where

$$A_2(x) = -2^{-1}kc_0(\xi_1^2(x) + k^{-2}(\psi_1'(x))^2) + kc_0\eta_1(x)V_1(x),$$

$$B_2(x) = -\eta_1(x)\xi_1(x) - 2^{-1}(kc_0)^{-1}bH_2(x);$$

— the system of equations of the 3rd approximation

$$\xi_3(x) = k^{-1}W\psi_3(x), \qquad (Rkc_0 + b)\eta_3(x) - b\psi_3(x) - kc_0\xi_3(x) - kc_0c_1\xi_1(x) = A_3(x),$$

$$-\eta_3(x) + \psi_3(x) - c_1\eta_1(x) = B_3(x), \qquad V_2(x) = k^{-2}(-\psi_2''(x) - \psi_1'(x)\xi_1'(x) + \psi_1''(x)\xi_1(x)),$$

where

$$A_3(x) = -kc_0(\xi_1(x)\xi_2(x) + k^{-2}\psi_1'(x)\psi_2'(x)) + kc_0(\eta_2(x)V_1(x) + \eta_1(x)V_{12}(x))$$

$$- 2^{-1}k^{-1}c_0H_2(x)\xi_1''(x) + 2^{-1}k^{-2}bH_2(x)\psi_1''(x),$$

$$B_3(x) = -\eta_2(x)\xi_1(x) - \eta_1(x)\xi_2(x) + 2^{-1}k^{-2}H_2(x)\psi_1''(x).$$

In each of these system of equation, the basic unknown functions should obey the periodicity and zero mean conditions. The solution of the system of equations of the 1st approximation is found by the formulas

$$\eta_1(x) = \cos(kx), \qquad \psi_1(x) = \cos(kx), \qquad \xi_1(x) = R\cos(kx).$$

The systems of successive approximations is solved using the following procedure. The solution of the system of equations of the 2nd and 3rd approximations starts with determining the functions $H_2(x)$, $V_1(x)$, $V_2(x)$, $A_2(x)$, $A_3(x)$, $B_2(x)$, and $B_3(x)$.

For the 2nd approximation, we obtain the trigonometric representations

$$H_2(x) = 2^{-1} + 2^{-1}\cos(2kx), \qquad V_1(x) = \cos(kx),$$
$$A_2(x) = 2^{-1}A_{20} + A_{21}\cos(2kx), \qquad B_2(x) = 2^{-1}B_{20} + B_{21}\cos(2kx), \tag{5}$$

where

$$A_{20} = -2^{-2}(R^2-1)kc_0, \qquad A_{21} = -2^{-2}(R^2-3)kc_0,$$
$$B_{20} = -2^{-1}(kc_0)^{-1}(2Rkc_0+b), \qquad B_{21} = 2^{-1}B_{20}. \tag{6}$$

The solution of the equations of the 2nd approximation is sought in a similar form:

$$\eta_2(x) = \eta_{21}\cos(2kx), \qquad \psi_2(x) = \psi_{21}\cos(2kx), \qquad \xi_2(x) = \xi_{21}\cos(2kx). \tag{7}$$

Then, the periodicity and zero mean conditions are satisfied automatically. Substitution of (5) and (7) into the system of equations of the 2nd approximation yields an algebraic system for the coefficients of these representations and P_1 and Q_1, whose solution gives

$$P_1 = -2^{-1}A_{20}, \qquad Q_1 = -2^{-1}B_{20}, \qquad \eta_{21} = -(kc_0)^{-1}R(A_{21} + (Rkc_0+b)B_{21}) - B_{21},$$
$$\psi_{21} = \eta_{21} + B_{21}, \qquad \xi_{21} = 2\coth(2kh)\psi_{21}.$$

The values of the constants P_1 and Q_1 and the coefficients of the trigonometric polynomials are calculated using equalities (6):

$$P_1 = 2^{-2}(R^2-1)kc_0, \qquad Q_1 = 2^{-2}(kc_0)^{-1}(2Rkc_0+b),$$
$$\eta_{21} = 2^{-2}(kc_0)^{-2}(R(3R^2-1)(kc_0)^2 + (3R^2+1)bkc_0 + Rb^2),$$
$$\psi_{21} = 2^{-2}(kc_0)^{-2}R(3(R^2-1)(kc_0)^2 + 3Rbkc_0 + b^2), \tag{8}$$
$$\xi_{21} = 2^{-1}(kc_0)^{-2}(R^2+1)(3(R^2-1)(kc_0)^2 + 3Rbkc_0 + b^2).$$

The solution of the system of equations for the 3rd approximation is found similarly. We obtain the trigonometric representations

$$A_3(x) = A_{30}\cos(kx) + A_{31}\cos(3kx), \qquad B_3(x) = B_{30}\cos(kx) + B_{31}\cos(3kx),$$
$$\eta_3(x) = \eta_{31}\cos(3kx),$$
$$\psi_3(x) = \psi_{30}\cos(kx) + \psi_{31}\cos(3kx), \qquad \xi_3(x) = \xi_{30}\cos(kx) + \xi_{31}\cos(3kx).$$

The computational formulas for c_1 and for the coefficients of these representations are as follows:

$$c_1 = -(2Rkc_0+b)^{-1}(A_{30} + (Rkc_0+b)B_{30}),$$
$$A_{30} = 2^{-3}(8kc_0\psi_{21} - 5Rkc_0 + 4kc_0\eta_{21} - 4Rkc_0\xi_{21} - 3b),$$
$$A_{31} = 2^{-3}(4kc_0\eta_{21} + 24kc_0\psi_{21} - 4Rkc_0\xi_{21} + R - b),$$
$$B_{30} = -2^{-3}(3 + 4(R + (kc_0)^{-1}b)\eta_{21}), \qquad B_{31} = B_{30} + 2^{-2},$$
$$\psi_{30} = c_1 + B_{30}, \qquad \xi_{30} = R\psi_{30},$$
$$\eta_{31} = -(kc_0)^{-1}D_3^{-1}(A_{31} + (Rkc_0+b)B_{31}) - B_{31},$$
$$\psi_{31} = \eta_{31} + B_{31}, \qquad \xi_{31} = 3\coth(3kh)\psi_{31}, \qquad D_3 = 2^3R(3R^2+1)^{-1}.$$

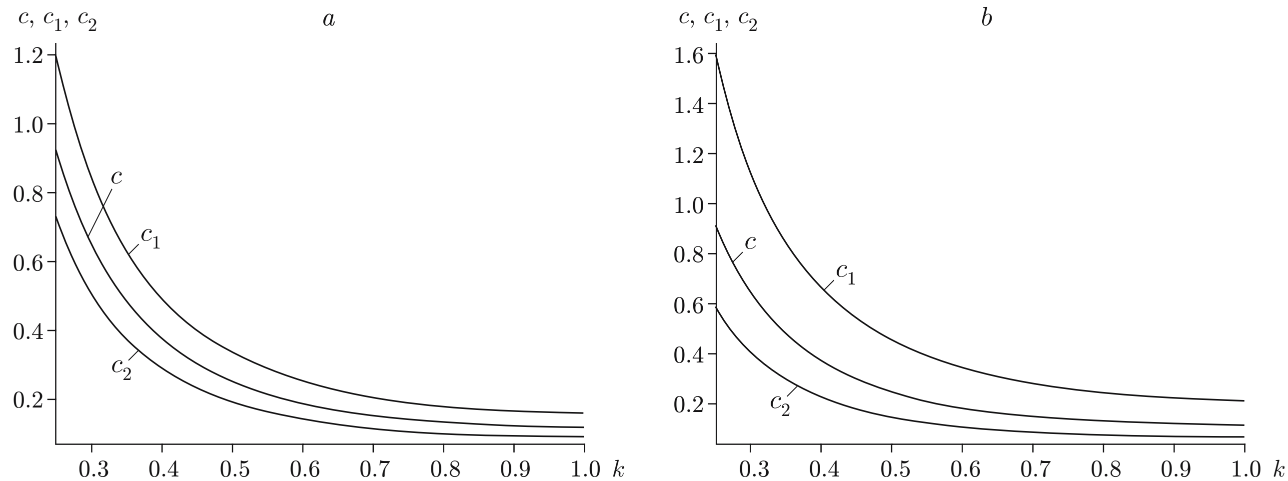

Fig. 1. Nonlinear correction to the wave velocity versus the wavenumber k for Stokes waves (velocity c), upstream traveling waves (velocity c_1), and upstream traveling waves (velocity c_2) for $b = 0.5$ (a) and 1.0 (b).

Using equalities (6) and (8), we find the quantity c_1 and the values of the coefficients of the trigonometric polynomials:

$$c_1 = 2^{-3}(2R + b_0)^{-1}(R(9R^4 - 10R^2 + 9) + 2(9R^4 - 2R^2 + 1)b_0$$
$$+ 3R(5R^2 + 1)b_0^2 + 2(3R^2 + 1)b_0^3 + Rb_0^4), \tag{9}$$

where $b_0 = b/(kc_0)$.

Using formula (9), we obtain nonlinear dispersion relations for both types of waves:

$$c^{(i)} = c_0^{(i)}(1 + 2^{-3}(2R + b_0)^{-1}(R(9R^4 - 10R^2 + 9) + 2(9R^4 - 2R^2 + 1)b_0$$
$$+ 3R(5R^2 + 1)b_0^2 + 2(3R^2 + 1)b_0^3 + Rb_0^4)(ka)^2)$$

(for $i = 1$, the wave moves upstream, and for $i = 2$ it moves downstream). In the case $b = 0$, these relations become the Stokes dispersion relation [5]

$$c = \sqrt{g(kR)^{-1}}\,(1 + 2^{-4}(9R^4 - 10R^2 + 9)(ka)^2).$$

The results of calculation of the nonlinear correction to the velocity of steady-state waves in the flow and its comparison with the Stokes waves velocity are given in Fig. 1 and lead to the following conclusions:

— in the presence of shear flow, the absolute values of the velocity for both (upstream and downstream traveling) waves increase;

— an increase in the flow gradient results in an increase in the absolute values of the wave velocity;

— the effect of the flow increases in the long-wavelength region and decreases in the short-wavelength region.

Conclusions. The procedure used here has the following features. The initial mathematical formulation of the problem uses a time-dependent stream function rather than the velocity potential, as was done previously (beginning with Stokes), which simplifies the boundary conditions. The reduction of the two-dimensional problem to a one-dimensional model significantly simplified the solution procedure and yielded the main results in compact form.

The method proposed here can be used to solve problems of the structure and characteristics of steady-state nonlinear surface and internal gravity waves in a stratified fluid, whose layers move in the horizontal direction with a linear mean-velocity profile in each layer.

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